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THE ALLOCATION OF RESOURCES IN
STEADY-STATE UNBALANCED GROWTH

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With few exceptions, modern growth models are models of steady-state and balanced[1] growth of homogeneous consumption and capital stock, hence miss imbalance [1], [6] as well as the allocation of resources.

To allow for imbalance, a growth model needs at least two goods. But to allow for the allocation of resources, the two goods cannot be the consumers' good and the producers' good found in the usual [5] two-sector growth models. With only one consumers' good, such models are still models of homogeneous consumption, permitting no substitution among consumers' goods and asking no question, hence offering no answer, concerning the allocation of consumption

expenditure among consumers goods. With only one producers' good such models are still models of homogeneous capital stock, permitting no substitution among producers' goods and asking no question, hence offering no answer, concerning the allocation of investment expenditure among producers' goods.

We wish to build the simplest possible growth model of heterogeneous consumption as well as capital stock, thus allowing for the full allocation of resources. To do that we assume <u>each</u> of our two goods to serve interchangeably as a consumers' or as a producers' good: The physical output of the jth good is X_j where $j = 1, 2$. The jth good is produced from labor L_j and two immortal capital stocks S_{ij} where $i = 1, 2$. There are, then, four capital stocks S_{ij} and four investments I_{ij} in our model. Between two such industries we specify a fourfold interaction:

The two industries compete in their demand for labor. In the

labor market they must pay the same money wage rate w, a parameter. Goods prices P_j are variables, hence the real wage rate w/P_j is also a variable.

The two industries compete in their demand for investment goods. In the market for the jth good they must pay the same price P_j. A firm producing the jth good and setting aside part of its own output for investment I_{jj} should charge itself the price P_j as an opportunity cost.

The two industries compete in their demand for money capital. In the money-capital market the capitalist-entrepreneurs allocate their savings between the two industries such as to maximize the present worth of all their future profits.

The two industries compete in their supply of consumers' goods. In the consumers' goods market the two goods are good, but not perfect, substitutes, and each consumer has a taste for both of them.

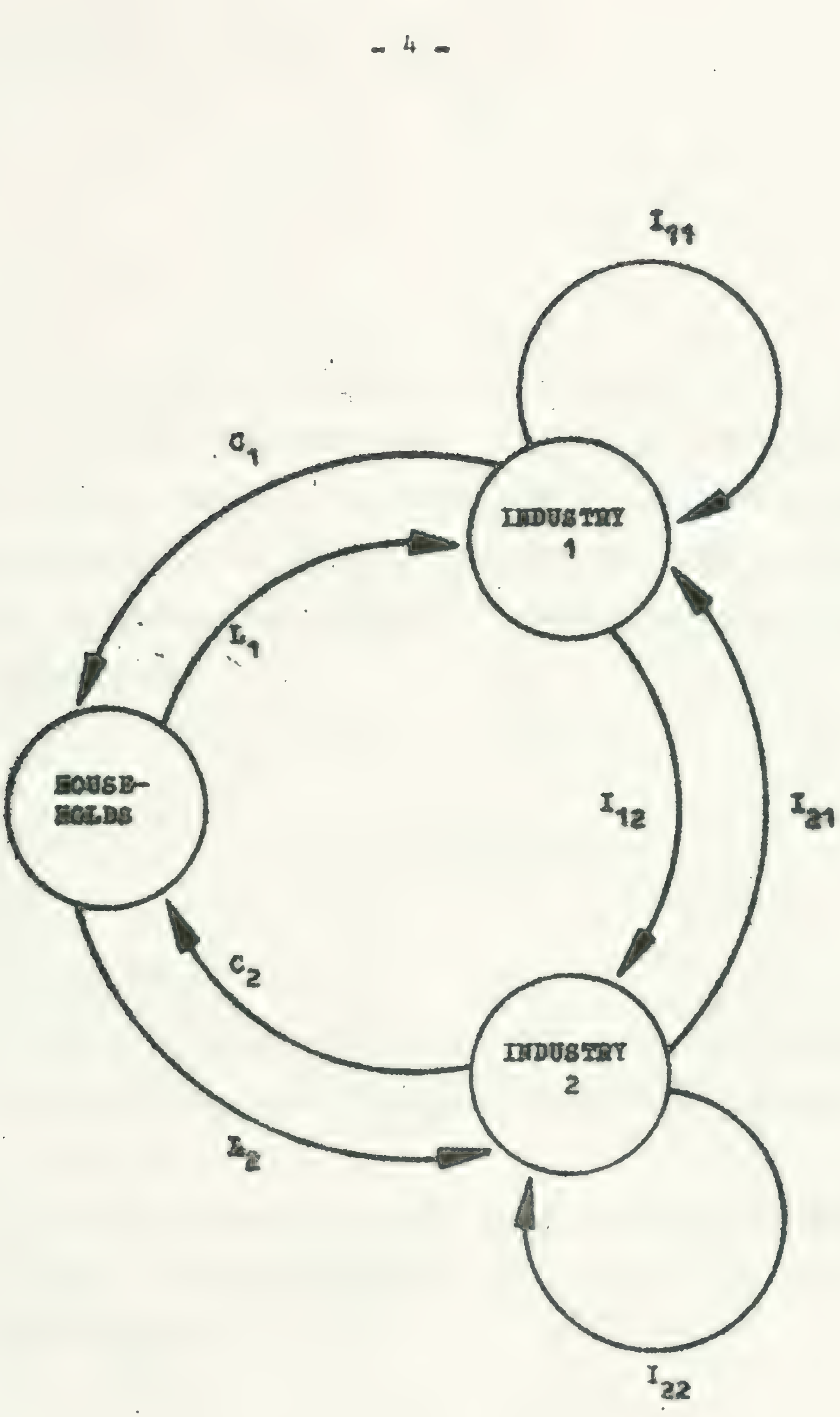

FIGURE 1. THE EIGHT PHYSICAL FLOWS

Figure 1 shows all physical flows in our model. Section I defines variables and parameters. Section II specifies the model mathematically. Section III finds the equilibrium solutions for proportionate rates of growth. Section IV finds the equilibrium solutions for levels of variables. Certain proofs are banished to two appendices.

I. NOTATION

Variables

C $\equiv$ consumption

ϕ $\equiv$ function to be maximized by the Lagrange-multiplier method

g_v $\equiv$ proportionate rate of growth of variable v where $v \equiv C$, I, L, P, S, X, and Y

I_{ij} $\equiv$ investment of output of ith industry in jth industry

κ_{ij} $\equiv$ physical marginal productivity of capital stock S_{ij}

L $\equiv$ labor employed

P ≡ price of good

S_{ij} ≡ jth industry's physical capital stock of ith industry's good

U ≡ utility

W ≡ wage bill

X ≡ physical output

Y ≡ national money income

Z ≡ profits bill

ζ ≡ present worth of all future profits bills

Parameters

A ≡ exponent of individual utility function

α, β ≡ exponents of production function

c ≡ propensity to consume national money income

e ≡ Euler's number, the base of natural logarithms

F ≡ available labor force

g_p ≡ proportionate rate of growth of parameter p where $p \equiv F, M,$

and w

λ $\equiv$ Lagrange multiplier

M $\equiv$ multiplicative factor of production function

N $\equiv$ multiplicative factor of individual utility function

r $\equiv$ discount rate applied by capitalist-entrepreneurs

w $\equiv$ money wage rate

The parameters listed are stationary except F, M, and w, whose growth rates g_F, g_M, and g_w are stationary.

The symbol π_i to be defined in Section II; h_j in Section IV, 2; ρ, m, n, and μ_i in Section IV, 3; ν_{ij} and ξ_i in Section IV, 5; and ψ in Appendix I all stand for agglomerations of parameters and variables. Symbols t and τ are time coordinates. Subscripts i = 1, 2 and j = 1, 2 refer to industry number. All flow variables refer to the instantaneous rate of that variable measured on a per annum basis.

II. THE EQUATIONS OF THE MODEL

17 variable growth rates are listed in Section I, i. e. two growth rates of each of C_i, L_i, P_i, and X_i; four growth rates of each of I_{ij} and S_{ij}; and one growth rate of Y. To all apply the definition

(1) through (17) $$g_v \equiv \frac{dv}{dt} \frac{1}{v}$$

Define investment as the derivative of capital stock with respect to time

(18) through (21) $$I_{ij} \equiv \frac{dS_{ij}}{dt}$$

Let the jth industry apply the Cobb-Douglas production function

(22) $$X_1 = M_1 L_1^{\alpha_1} S_{11}^{\beta_{11}} S_{21}^{\beta_{21}}$$

(23) $$X_2 = M_2 L_2^{\alpha_2} S_{12}^{\beta_{12}} S_{22}^{\beta_{22}}$$

where $0 < \alpha_j < 1$; $0 < \beta_{ij} < 1$; $\alpha_1 + \beta_{11} + \beta_{21} = 1$; $\alpha_2 + \beta_{12} + \beta_{22} = 1$; and $M_j > 0$. In each industry let profit maximization under pure competition equalize real wage rate and physical marginal productivity of labor:

(24), (25) $$\frac{w}{P_j} = \frac{\partial X_j}{\partial L_j} = \alpha_j \frac{X_j}{L_j}$$

Physical marginal productivities of capital at time t are

(26) through (29) $$\kappa_{ij}(t) \equiv \frac{\partial X_j(t)}{\partial S_{ij}(t)} = \beta_{ij} \frac{X_j(t)}{S_{ij}(t)}$$

Multiply (26) through (29) by price of output of jth industry $P_j(t)$ to find value marginal productivities of capital at time t. Define money profits earned at time t on each physical unit of capital stock $S_{ij}(t)$ as its value marginal productivity. Then multiply by $S_{ij}(t)$ to find money profits earned at time t on capital stock $S_{ij}(t)$. Sum over i = 1, 2 and define the outcome as money profits earned at time t on whatever capital stock exists at that time in the entire jth industry:

(30), (31) $$Z_j(t) \equiv \sum_{i=1}^{2} \kappa_{ij}(t)P_j(t)S_{ij}(t) = P_j(t)X_j(t) \sum_{i=1}^{2} \beta_{ij}$$

Sum over j = 1, 2 and define the outcome as money profits earned at time t on whatever capital stock exists at that time in the entire economy:

$$Z(t) \equiv \sum_{j=1}^{2} Z_j(t) \tag{32}$$

As seen from the present time τ this profits bill is $Z(t)e^{-r(t - \tau)}$ where e is Euler's number, the base of natural logarithms, and r is the discount rate applied by the capitalist-entrepreneurs. Finally integrate this over $t = \tau$ through ∞ and define the outcome as the present worth of all future profits bills

$$\zeta(\tau) \equiv \int_{\tau}^{\infty} Z(t)e^{-r(t - \tau)}dt \tag{33}$$

Now let capitalist-entrepreneurs use their control variable I_{ij} to optimize the allocation of their capital stock

S_{ij} within as well as between industries. Within the jth industry they act as stockholders optimizing S_{ij} where i = 1, 2 by appointing the right managers. Between industries they act as stockholders optimizing S_{ij} where j = 1, 2 by purchasing stock in the right industry. "Optimizing" in what sense? In the sense that

$$\zeta(\tau) = \text{maximum} \tag{34}$$

Under full employment, available labor force must equal the sum of labor employed by the two industries:

$$F = \sum_{i=1}^{2} L_i \tag{35}$$

Define the wage bill as the money wage rate <u>times</u> employment:

(36) $$W \equiv w \sum_{i=1}^{2} L_i$$

Define national money income as the sum of the wage bill and the profits bill:

(37) $$Y \equiv W + Z$$

Let all persons have the same utility function. Let the utility function of the kth person be

$$U_k = N C_{1k}^{A_1} C_{2k}^{A_2}$$

where $0 < A_i < 1$ and $N > 0$. Let there be s persons, and let the kth person's money income be Y_k where

$$\sum_{k=1}^{s} Y_k = Y$$

Let all persons spend the fraction c, where $0 < c < 1$, of their money income. Then the budget constraint of the kth person is

$$cY_k = P_1C_{1k} + P_2C_{2k}$$

Maximize the kth person's utility subject to his budget constraint and find his two demand functions. Then add the s individual demand functions for each good and find the two Graham [2] aggregate demand functions

$$C_i = \pi_i Y/P_i \qquad (38), (39)$$

where

$$\pi_i = \frac{cA_i}{A_1 + A_2}$$

Industry output equilibrium requires the output of the ith industry to equal the sum of consumption and investment demand for it, or inventory would either accumulate or be depleted:

(40), (41) $$X_i = C_i + \sum_{j=1}^{2} I_{ij}$$

III. SOLUTIONS FOR PROPORTIONATE RATES OF GROWTH

Our system (1) through (41) possesses the following set of steady-state solutions for its equilibrium proportionate rates of growth:

(42), (43) $$g_{Ci} = g_{Xi}$$

Industry output equilibrium requires that [illegible] of the ith

industry be equal [illegible]

[illegible]

$$ [illegible] \qquad (10), (11) $$

III. [illegible]

Our system (1) through [illegible]

[illegible] (14) [illegible]

$$ [illegible] \qquad (12), (13) $$

(44) through (47) $g_{Iij} = g_{Xi}$

(48), (49) $g_{Li} = g_F$

(50) $$g_{P1} = g_w - \frac{(1 - \beta_{22})g_{M1} + \beta_{21}g_{M2}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}$$

(51) $$g_{P2} = g_w - \frac{(1 - \beta_{11})g_{M2} + \beta_{12}g_{M1}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}$$

(52) through (55) $g_{Sij} = g_{Xi}$

(56) $$g_{X1} = \frac{(1 - \beta_{22})g_{M1} + \beta_{21}g_{M2}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}} + g_F$$

(57) $$g_{X2} = \frac{(1 - \beta_{11})g_{M2} + \beta_{12}g_{M1}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}} + g_F$$

(58) $$g_Y = g_F + g_w$$

To see that it does, the reader should take derivatives with respect to time of all equations (18) through (41) except (26) through (29) and (33), (34). He should then use definitions (1) through (17), insert solutions (42) through (58), and convince himself that each equation is satisfied.

We defined balanced growth as identical proportionate rates of growth of physical output for all goods. According to our solutions (42) through (58), is our steady-state growth balanced or unbalanced?

Growth does spill over from one industry to the other. For example, according to (44) through (47) a more rapidly growing industry i would transmit some of its growth to a more slowly growing industry j investing in the ith industry's good. But the spillover is normally not enough to generate balanced growth. Use (56), (57), and the assumptions that $\alpha_1 + \beta_{11} + \beta_{21} = 1$ and $\alpha_2 + \beta_{12} + \beta_{22} = 1$ to find that

$$g_{X1} \gtreqless g_{X2} \text{ implies } g_{M1}/g_{M2} \gtreqless \alpha_1/\alpha_2,$$

respectively. Or in English: The first industry's physical output may grow more rapidly than that of the second industry for two and only[2] two reasons, i. e., first if everything else being equal the

first industry has more rapid technological progress g_{Mi} than the second industry, second, if everything else being equal the physical output of the first industry has a lower labor elasticity α_i than that of the second industry: The less labor-sensitive industry is less hampered by the fact that under technological progress labor force is growing less rapidly than physical capital stocks.

It does, however, follow from (50), (51), (56), and (57) that unlike physical outputs X_i, industry revenues P_iX_i will grow at the same proportionate rate $g_F + g_W$.

IV. SOLUTIONS FOR LEVELS

So much for proportionate rates of growth. Let us now turn to the allocation of resources and solve for the allocation of savings between industries; the levels of industry revenues; employments; national money income; physical outputs; prices; physical

capital stocks and their physical marginal productivities; consumption; and income distribution.

1. Saving Equals Investment

Use (24), (25), and (36) to see that $W = \alpha_1 P_1 X_1 + \alpha_2 P_2 X_2$, and (30) through (32) to see that $Z = (\beta_{11} + \beta_{21})P_1 X_1 + (\beta_{12} + \beta_{22})P_2 X_2$, hence national income equals national output:

$$Y = P_1 X_1 + P_2 X_2 \tag{59}$$

Multiply (40) and (41) by P_1 and P_2, respectively, insert (38), (39), and (59) and find saving to equal investment:

$$(1 - c)Y = P_1(I_{11} + I_{12}) + P_2(I_{21} + I_{22}) \tag{60}$$

2. Present-Worth Maximization

Subject to the constraint (60) let the capitalist-entrepreneurs use their control variable I_{ij} to optimize the allocation of their capital stock S_{ij} within as well as between industries. "Optimize" in what sense? In the sense of maximizing the present worth $\zeta(\tau)$ of all future profits bills in accordance with (34). Using (30) through (33) we write present worth as

$$\zeta(\tau) = \int_{\tau}^{\infty} [(\beta_{11} + \beta_{21})P_1(t)X_1(t) + (\beta_{12} + \beta_{22})P_2(t)X_2(t)]e^{-r(t - \tau)}dt$$

Let it be foreseen by the capitalist-entrepreneurs that prices are growing in accordance with our steady-state solutions (50) and (51), hence

$$P_j(t) = e^{g_{Pj}(t - \tau)} P_j(\tau)$$

and that outputs are growing in accordance with our steady-state solutions (56) and (57), hence

$$X_j(t) = e^{g_{Xj}(t - \tau)} X_j(\tau)$$

Consequently we may take prices and outputs outside the integral sign and write present worth as

$$\zeta(\tau) = (\beta_{11} + \beta_{21})P_1(\tau)X_1(\tau)\int_\tau^\infty e^{(g_{P1} + g_{X1} - r)(t - \tau)}dt +$$

$$(\beta_{12} + \beta_{22})P_2(\tau)X_2(\tau)\int_\tau^\infty e^{(g_{P2} + g_{X2} - r)(t - \tau)}dt$$

Since in this expression all variables refer to the same time

τ, we may purge it of τ. Use (50), (51), (56), and (57) to see that $g_{Pj} + g_{Xj} = g_F + g_w$. Assume that $g_F + g_w < r$, then integrate:

$$\zeta = \frac{(\beta_{11} + \beta_{21})P_1X_1 + (\beta_{12} + \beta_{22})P_2X_2}{r - (g_F + g_w)}$$

Inserting (30) through (32) into this we find the simple relationship between profits and present worth under steady-state growth:

$$Z = [r - (g_F + g_w)]\zeta \tag{61}$$

Maximizing present worth ζ subject to the constraint (60) is most easily done by using a Lagrange multiplier: Define a new function to be maximized

$$\phi \equiv \zeta + \lambda[(1 - c)Y - P_1(I_{11} + I_{12}) - P_2(I_{21} + I_{22})]$$

What to do with Y? Insert (61) into (37), insert the outcome into ϕ and write the latter

$$\phi = \{1 + \lambda(1 - c)[r - (g_F + g_w)]\}\zeta + \lambda(1 - c)W - \lambda[P_1(I_{11} + I_{12}) + P_2(I_{21} + I_{22})] \tag{62}$$

The first four first-order conditions for a maximum ϕ are

$$\frac{\partial \phi}{\partial I_{ij}} = h_j \frac{\partial X_j}{\partial I_{ij}} - \lambda P_i = 0 \tag{63}$$

where

$$h_j \equiv \frac{\{1 + \lambda(1 - c)[r - (g_F + g_W)]\}(\beta_{1j} + \beta_{2j})P_j}{r - (g_F + g_W)}$$

Now according to the production functions (22) and (23), output X_j is a function of capital stock S_{ij} rather than of investment I_{ij}. But according to (1) through (21)

$$S_{ij} \equiv I_{ij}/g_{Sij} \tag{64}$$

where our steady-state growth, as specified by (52) through (57), permits us to express g_{Sij} solely in terms of parameters. Inserting (64) into the production functions (22) and (23) we find

$$\frac{\partial X_j}{\partial I_{ij}} = \beta_{ij} \frac{X_j}{I_{ij}} \tag{65}$$

and write the first-order conditions as

(66) through (69)

$$\beta_{11}(\beta_{11} + \beta_{21})X_1/I_{11} = \beta_{12}(\beta_{12} + \beta_{22})P_2X_2/(P_1I_{12})$$

$$= \beta_{21}(\beta_{11} + \beta_{21})P_1X_1/(P_2I_{21}) = \beta_{22}(\beta_{12} + \beta_{22})X_2/I_{22}$$

$$= \frac{\lambda[r - (g_F + g_W)]}{1 + \lambda(1 - c)[r - (g_F + g_W)]}$$

That the second-order conditions are satisfied is demonstrated

in Appendix I.

3. Solving for Industry Revenues P_jX_j

Use the first-order conditions (66) through (68) to express I_{12} in terms of I_{11} and I_{21} in terms of I_{22}. Insert the results into (40) and (41). Insert (59) into (38) and (39). Insert the results into (40) and (41). Divide (40) by $\beta_{11}(\beta_{11} + \beta_{21})X_1$ and (41) by $\beta_{22}(\beta_{12} + \beta_{22})X_2$, deduct (41) from (40), and again use the first-order conditions (66) through (68). Now define

(70) $$\rho \equiv (P_1X_1/(P_2X_2)$$

rearrange, and write the quadratic

(71) $$\rho^2 + m\rho + n = 0$$

where m and n are the following agglomerations of taste and

technology parameters

$$m \equiv \frac{(\beta_{12} + \beta_{22})[\beta_{12}\pi_2 + \beta_{22}(1 - \pi_1)] - (\beta_{11} + \beta_{21})[\beta_{11}(1 - \pi_2) + \beta_{21}\pi_1]}{(\beta_{11} + \beta_{21})[\beta_{11}\pi_2 + \beta_{21}(1 - \pi_1)]}$$

$$n \equiv - \frac{(\beta_{12} + \beta_{22})[\beta_{12}(1 - \pi_2) + \beta_{22}\pi_1]}{(\beta_{11} + \beta_{21})[\beta_{11}\pi_2 + \beta_{21}(1 - \pi_1)]}$$

The quadratic has the two roots

$$\rho = - m/2 \pm \sqrt{(m/2)^2 - n}$$

We have assumed that $0 < A_i < 1$, $0 < \beta_i < 1$, and $0 < c < 1$, hence $n < 0$. Now regardless of the sign of m, $0 \leq (m/2)^2$, hence

$$0 \leq (m/2)^2 < (m/2)^2 - n$$

Two things follow. First, from $0 < (m/2)^2 - n$ it follows that both roots are real. Second, from $(m/2)^2 < (m/2)^2 - n$ it follows that regardless of the sign of m, the first root is positive and the second negative. We reject the latter and are left with

$$\rho = -m/2 + \sqrt{(m/2)^2 - n} \tag{72}$$

Use (24), (25), (35), and (36) to find

$$\alpha_1 P_1 X_1 + \alpha_2 P_2 X_2 = wF$$

Take this together with (70) and find

(73), (74) $$P_iX_i = \mu_i wF$$

where

$$\mu_1 \equiv \rho/(\alpha_1\rho + \alpha_2)$$

$$\mu_2 \equiv 1/(\alpha_1\rho + \alpha_2)$$

4. Solving for Employments L_i and Income Y

Use (24), (25), (73), (74) to solve for employments

(75), (76) $$L_i = \alpha_i\mu_i F$$

Insert (73) and (74) into (59) and solve for national money income

$$Y = (\mu_1 + \mu_2)wF \tag{77}$$

5. Solving for Physical Outputs X_j

Let us begin by finding four investment-output ratios. Again use the first-order conditions (66) through (68) to express I_{12} in terms of I_{11}. Insert the result into (40), insert (59) into (38), and insert the result into (40). Divide (40) by X_1. Use a similar procedure upon (41) and find the four ratios

$$I_{11}/X_1 = \nu_{11} \equiv \frac{1 - \pi_1 - \pi_1/\rho}{1 + \beta_{12}(\beta_{12} + \beta_{22})/[\rho\beta_{11}(\beta_{11} + \beta_{21})]} \tag{78}$$

$$I_{12}/X_1 = \nu_{12} \equiv \frac{1 - \pi_1 - \pi_1/\rho}{1 + \rho\beta_{11}(\beta_{11} + \beta_{21})/[\beta_{12}(\beta_{12} + \beta_{22})]} \tag{79}$$

(80) $$I_{21}/X_2 = \nu_{21} \equiv \frac{1 - \pi_2 - \pi_2\rho}{1 + \beta_{22}(\beta_{12} + \beta_{22})/[\rho\beta_{21}(\beta_{11} + \beta_{21})]}$$

(81) $$I_{22}/X_2 = \nu_{22} \equiv \frac{1 - \pi_2 - \pi_2\rho}{1 + \rho\beta_{21}(\beta_{11} + \beta_{21})/[\beta_{22}(\beta_{12} + \beta_{22})]}$$

Apply (64) to (78) through (81) and find

(82) $$S_{ij} = X_i\nu_{ij}/g_{Sij}$$

Insert (82) and our solutions (75) and (76) into the production functions (22) and (23), arrive at two equations in the two unknowns X_j, solve them, and find

$$X_1 = (\xi_1^{1 - \beta_{22}} \xi_2^{\beta_{21}})^{\frac{1}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}} F \tag{83}$$

$$X_2 = (\xi_1^{\beta_{12}} \xi_2^{1 - \beta_{11}})^{\frac{1}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}} F \tag{84}$$

where

$$\xi_1 \equiv M_1(\alpha_1\mu_1)^{\alpha_1}(\nu_{11}/g_{S11})^{\beta_{11}}(\nu_{21}/g_{S21})^{\beta_{21}}$$

$$\xi_2 \equiv M_2(\alpha_2\mu_2)^{\alpha_2}(\nu_{12}/g_{S12})^{\beta_{12}}(\nu_{22}/g_{S22})^{\beta_{22}}$$

The reader may convince himself that (83) and (84) are indeed growing at the rates (56) and (57) said they should be.

6. Solving for Prices P_j

Divide our revenue solutions (73) and (74) by our physical output solutions (83) and (84), respectively, and find

$$(85) \qquad P_1 = (\xi_1^{1 - \beta_{22}} \xi_2^{\beta_{21}})^{-\frac{1}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}} w\mu_1$$

$$(86) \qquad P_2 = (\xi_1^{\beta_{12}} \xi_2^{1 - \beta_{11}})^{-\frac{1}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}} w\mu_2$$

Similarly the reader may convince himself that (85) and (86) are indeed growing at the rates (50) and (51) said they should be.

7. Capital Stocks S_{ij} and their Marginal Productivities κ_{ij}

With (83) and (84) inserted into it, (82) will be a solution for physical capital stocks S_{ij}. With (82) inserted into them, (26) through (29) will be solutions for the physical marginal productivities of capital

(87) $$\kappa_{ij} = \beta_{ij} g_{Sij} / \nu_{ij}$$

8. Consumption C_i and Income Distribution W and Z

With (77), (85), and (86) inserted into them, (38) and (39) will be solutions for consumption. With (35) inserted into it, (36) will be a solution for the wage bill

(88) $$W = wF$$

With (73) and (74) inserted into them, (30) through (32) will generate the profits bill

$$(89) \qquad Z = [(\beta_{11} + \beta_{21})\mu_1 + (\beta_{12} + \beta_{22})\mu_2]wF$$

With (89) inserted into it, (61) will be a solution for present worth.

9. Properties of Solutions

We have now solved for the levels of all variables. Our solutions (78) through (81) for the investment-output ratios and (87) for the physical marginal productivities of capital are stationary. All other solutions for levels are nonstationary, because they contain one or more of our three nonstationary

parameters, i. e. available labor force F, the multiplicative factor M_i of the production functions, and the money wage rate w.

Are our solutions real and positive? Section IV, 3 found both roots ρ to be real and found one to be positive, the other negative. All solutions (73) through (89), then, are real. Rejecting the negative root we find solutions (73) through (77), (88), and (89) to be obviously positive. Less obviously, so are solutions (78) through (87), as demonstrated in our Appendix II.

APPENDIX I

SECOND-ORDER CONDITIONS FOR A MAXIMUM OF EQUATION (62)

Write the bordered Hessian

$$(90) \quad H \equiv \begin{vmatrix} \dfrac{\partial^2\phi}{\partial I_{11}^2} & \dfrac{\partial^2\phi}{\partial I_{11}\partial I_{12}} & \dfrac{\partial^2\phi}{\partial I_{11}\partial I_{21}} & \dfrac{\partial^2\phi}{\partial I_{11}\partial I_{22}} & -P_1 \\ \dfrac{\partial^2\phi}{\partial I_{12}\partial I_{11}} & \dfrac{\partial^2\phi}{\partial I_{12}^2} & \dfrac{\partial^2\phi}{\partial I_{12}\partial I_{21}} & \dfrac{\partial^2\phi}{\partial I_{12}\partial I_{22}} & -P_1 \\ \dfrac{\partial^2\phi}{\partial I_{21}\partial I_{11}} & \dfrac{\partial^2\phi}{\partial I_{21}\partial I_{12}} & \dfrac{\partial^2\phi}{\partial I_{21}^2} & \dfrac{\partial^2\phi}{\partial I_{21}\partial I_{22}} & -P_2 \\ \dfrac{\partial^2\phi}{\partial I_{22}\partial I_{11}} & \dfrac{\partial^2\phi}{\partial I_{22}\partial I_{12}} & \dfrac{\partial^2\phi}{\partial I_{22}\partial I_{21}} & \dfrac{\partial^2\phi}{\partial I_{22}^2} & -P_2 \\ -P_1 & -P_1 & -P_2 & -P_2 & 0 \end{vmatrix}$$

The first derivatives $\partial\phi/\partial I_{ij}$ have already been taken and were of the form (63). It follows from that form that a good many of the second derivatives contained in our Hessian are zero: After inserting (64) into our production functions (22) and (23) we realize that X_j is a function of neither I_{ii} nor I_{ji} where $i \neq j$, hence

$$(91)\quad \frac{\partial X_j}{\partial I_{ii}} = \frac{\partial X_j}{\partial I_{ji}} = \frac{\partial^2 X_j}{\partial I_{ii}\partial I_{ij}} = \frac{\partial^2 X_j}{\partial I_{ji}\partial I_{ij}} = \frac{\partial^2 X_j}{\partial I_{ii}\partial I_{jj}} = \frac{\partial^2 X_j}{\partial I_{ji}\partial I_{jj}} = 0$$

$$(i \neq j)$$

But X_j <u>is</u> a function of I_{ij} and I_{jj}, hence

$$(92)\qquad \frac{\partial^2 X_j}{\partial I_{ij}\partial I_{jj}} = \frac{\partial^2 X_j}{\partial I_{jj}\partial I_{ij}} = \frac{\beta_{ij}\beta_{jj}X_j}{I_{ij}I_{jj}} \qquad (i \neq j)$$

(93) $$\frac{\partial^2 X_j}{\partial I_{ij}^2} = \beta_{ij}(\beta_{ij} - 1)\,\frac{X_j}{I_{ij}^2} \qquad (i = j \text{ or } i \neq j)$$

Apply (91), (92), and (93) to the Hessian (90). Then try to produce even more zero elements, making the Hessian easier to evaluate. Factor out $\beta_{11}h_1X_1/I_{11}$ from first row; $\beta_{12}h_2X_2/I_{12}$ from second row; $\beta_{21}h_1X_1/I_{21}$ from third row; and $\beta_{22}h_2X_2/I_{22}$ from fourth row, where h was defined as part of (63). Thereby the first four elements of the fifth column become

$$- P_1I_{11}/(\beta_{11}h_1X_1),$$

$$- P_1I_{12}/(\beta_{12}h_2X_2),$$

$$- P_2I_{21}/(\beta_{21}h_1X_1),$$

$$- P_2I_{22}/(\beta_{22}h_2X_2)$$

But according to the first-order conditions (66) through (69) those four values are all equal to $-1/\lambda$. Now factor out $1/I_{11}$ from first column, $1/I_{12}$ from second column, $1/I_{21}$ from third column, $1/I_{22}$ from fourth column, and $1/\lambda$ from fifth column.

If to each element of a row is added the corresponding element of another row, the determinant remains unchanged. So factor out (-1) from the first row and add to each element of it the corresponding element of the third row. Factor out (-1) from the second row and add to each element of it the corresponding element of the fourth row.

If to each element of a column is added the corresponding element of another column, the determinant remains unchanged. So add to each element of the third column the corresponding element of the first column. Add to each element of the fourth column the corresponding element of the second column.

By now the Hessian has been transformed into the following very tractable form:

$$H = \frac{\beta_{11}\beta_{12}\beta_{21}\beta_{22}h_1{}^2h_2{}^2X_1{}^2X_2{}^2}{I_{11}{}^2I_{12}{}^2I_{21}{}^2I_{22}{}^2\lambda} \times$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \beta_{11} & 0 & -\alpha_1 & 0 & -1 \\ 0 & \beta_{12} & 0 & -\alpha_2 & -1 \\ -P_1I_{11} & -P_1I_{12} & -P_1I_{11}-P_2I_{21} & -P_1I_{12}-P_2I_{22} & 0 \end{vmatrix}$$

$$= \frac{\beta_{11}\beta_{12}\beta_{21}\beta_{22}h_1{}^2h_2{}^2X_1{}^2X_2{}^2}{I_{11}{}^2I_{12}{}^2I_{21}{}^2I_{22}{}^2\lambda}[\alpha_2(P_1I_{11} + P_2I_{21}) + \alpha_1(P_1I_{12} + P_2I_{22})]$$

Is our Hessian positive, then? Appendix II will demonstrate that all solutions, including those for P_iI_{ij}, are positive. To see if λ is positive, write the fifth first-order condition $\partial\phi/\partial\lambda = 0$ and find it to be the constraint (60). Use (66) through (69) to write

$$P_1(I_{11} + I_{12}) + P_2(I_{21} + I_{22}) =$$

$$\frac{1 + \lambda(1 - c)[r - (g_F + g_w)]}{\lambda[r - (g_F + g_w)}[(\beta_{11} + \beta_{21})^2P_1X_1 + (\beta_{12} + \beta_{22})^2P_2X_2$$

Insert (59) and this into the constraint (60), rearrange, and

write the latter

$$\lambda = \frac{\psi}{(1 - c)(1 - \psi)[r - (g_F + g_W)]}$$

where

$$\psi \equiv \frac{(\beta_{11} + \beta_{21})^2 P_1 X_1 + (\beta_{12} + \beta_{22})^2 P_2 X_2}{P_1 X_1 + P_2 X_2}$$

It follows from $0 < \psi < 1$ that $\lambda > 0$, hence the Hassian (90) is positive. And now for its principal minors.

From the Hessian (90) remove successively fourth, third, and second column and row and obtain the bordered 4 × 4, 3 × 3, and 2 × 2 principal minors. Their values are respectively

$$-\frac{\beta_{11}\beta_{12}\beta_{21}h_1{}^2h_2X_1{}^2X_2}{I_{11}{}^2I_{12}{}^2I_{21}{}^2\lambda}[(1-\beta_{12})(P_1I_{11}+P_2I_{21})+(1-\beta_{11}-\beta_{21})P_1I_{12}]$$

$$\frac{\beta_{11}\beta_{12}h_1h_2X_1X_2}{I_{11}{}^2I_{12}{}^2\lambda}[(1-\beta_{12})P_1I_{11}+(1-\beta_{11})P_1I_{12}]$$

$$-\frac{\beta_{11}h_1X_1}{I_{11}{}^2\lambda}P_1I_{11}$$

The three values are negative, positive, and negative, respectively.

A P P E N D I X I I

SIGN OF SOLUTIONS (78) THROUGH (87)

Solutions (78) through (87) contain one of the factors ν_{ij}. Could those factors be nonpositive? To show that they cannot we prove that our positive root ρ has the following bounds:

$$\pi_1/(1 - \pi_1) < \rho < (1 - \pi_2)/\pi_2 \tag{94}$$

Take the first inequality of (94), insert (72), move the term $-(m/2)$ to the other side, and write the inequality

$$\sqrt{(m/2)^2 - n} > m/2 + \pi_1/(1 - \pi_1)$$

Square the inequality, multiply it by $(1 - \pi_1)^2$, and write it

$$- \pi_1^2 - m\pi_1(1 - \pi_1) - n(1 - \pi_1)^2 > 0$$

Now insert the definitions of m and n attached to (71), recall that $\pi_1 + \pi_2 = c$, rearrange, and find

$$(1 - c)[(\beta_{11} + \beta_{21})\beta_{11}\pi_1 + (\beta_{12} + \beta_{22})\beta_{12}(1 - \pi_1)] > 0$$

which it is under our assumptions about β_{ij} and π_i.

Then take the second inequality of (94), insert (72), move the term $-(m/2)$ to the other side, and write the inequality

$$\sqrt{(m/2)^2 - n} < m/2 + (1 - \pi_2)/\pi_2$$

Square the inequality, multiply it by $\pi_2{}^2$, and write it

$$(1 - \pi_2)^2 + m\pi_2(1 - \pi_2) + n\pi_2{}^2 > 0$$

Insert the definitions of m and n and find

$$(1 - c)[(\beta_{11} + \beta_{21})\beta_{21}(1 - \pi_2) + (\beta_{12} + \beta_{22})\beta_{22}\pi_2] > 0$$

which it is under our assumptions about β_{ij} and π_i.

Now that we have validated (94), take its first inequality, multiply it by $1 - \pi_1$, divide it by ρ, use the definitions (78) and (79) and find

$$\nu_{11} > 0 \qquad \nu_{12} > 0$$

Take the second inequality of (94), multiply it by π_2, use the definitions (80) and (81) and find

$$\nu_{21} > 0 \qquad \nu_{22} > 0$$

We conclude that (78) through (87) are indeed positive.

A P P E N D I X I I I

EMPIRICAL MEASUREMENT OF GROWTH IMBALANCE

Yotopoulos and Lau [6] have examined growth imbalance in 65 countries for the periods 1948-53, 1954-58, and 1950-60. In each country, six sectors were distinguished, i. e. agriculture, mining, manufacturing, construction, electricity-gas-water and "others," including transportation and communication, services, etc.

Modifying the Yotopoulos-Lau notation slightly to make it consistent with our own, let us define

E_i ≡ income elasticity of demand for output of ith sector

G ≡ proportionate rate of growth of gross domestic product in constant prices

g_{Xi} ≡ proportionate rate of growth of output of ith sector in constant prices

ω_i ≡ share in gross domestic product of value added by ith sector

Yotopoulos-Lau now applied two different concepts of imbalance. First, an index of Samuelson-Solow-von Neumann imbalance defined as

$$(95) \qquad V^* \equiv \frac{1}{G} \sqrt{\sum_{i=1}^{n} \omega_i (g_{Xi} - G)^2}$$

or, in English, the reciprocal of the national real growth rate times the square root of the weighted sum of the squared deviations of sectoral real growth rates from the national real growth rate.

For their entire sample of 65 countries, Yotopoulos-Lau found a rather strong negative correlation between the Samuelson-Solow -von Neumann index of imbalance and the national real growth rate; The coefficient of correlation was -0.322. They also found the most highly developed countries to have have the lowest index of

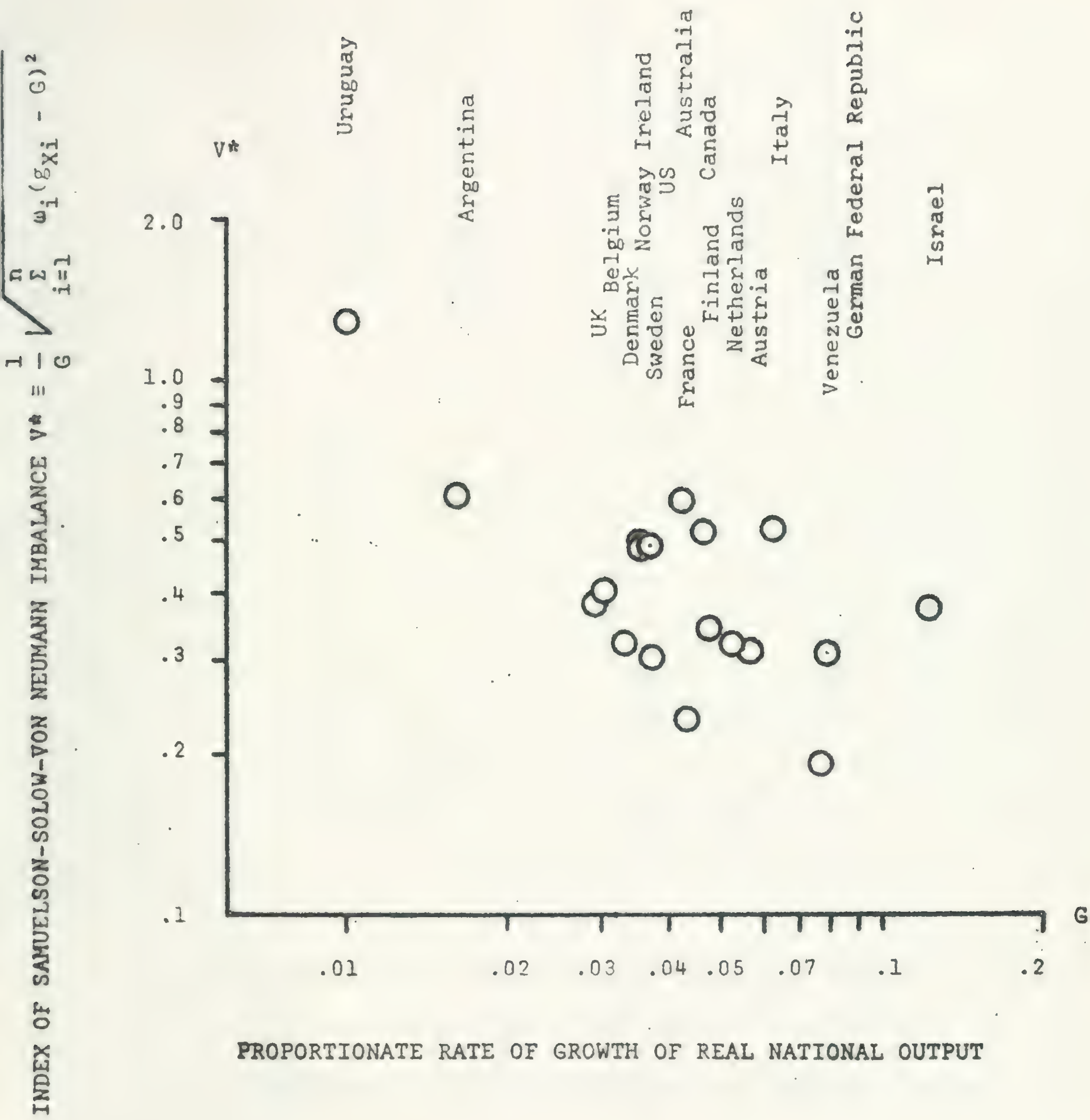

FIGURE 2. SAMUELSON-SOLOW-VON NEUMANN IMBALANCE IN 19 COUNTRIES 1950-60

Hans Brems, "Allocation of Resources in Steady-State Unbalanced Growth"

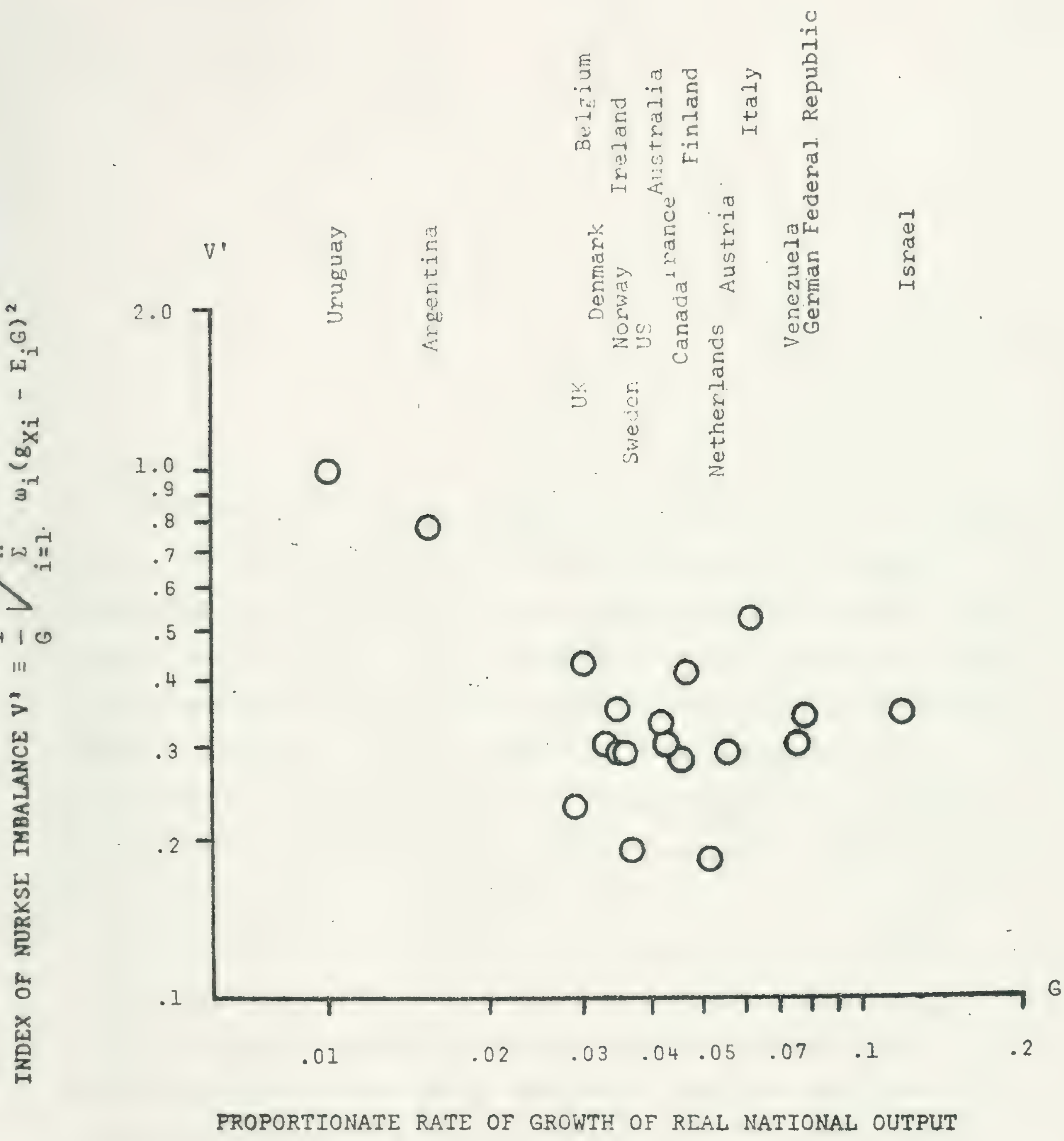

FIGURE 3. NURKSE IMBALANCE IN 19 COUNTRIES 1950-60

imbalance.

From the Yotopoulos-Lau sample of 65 countries our own Figure 2 has selected, for the period 1950-60, a much smaller sample consisting of the 19 capitalist countries which had, in 1958, a per capita income of $500 or more per annum. Figure 2 shows that even those countries still had a substantial Samuelson-Solow-von Neumann index of imbalance: Their square root of the weighted sum of squared deviations ranged from 0.19 (Venezuela) to 1.26 (Uruguay) of the national real growth rate, with the majority of the countries lying between 0.30 and 0.55 of that rate.

Could imbalance be explained by nonunitary sector income elasticities? Here it occurred to Yotopoulos-Lau to define a second index of imbalance removing from the imbalance concept those deviations which are caused by nonunitary sector income elasticities. That index they called a Nurkse imbalance index and defined it as

$$V' \equiv \frac{1}{G} \sqrt{\sum_{i=1}^{n} \omega_i (g_{Xi} - E_i G)^2} \tag{96}$$

or, in English, the reciprocal of the national real growth rate <u>times</u> the square root of the weighted sum of the squared deviations of sectoral real growth rates from the product of sector income elasticity and national real growth rate.

Now suppose that imbalance were fully explained by nonunitary sector income elasticities. Then the output of the ith sector would always be growing at the rate $g_{Xi} = E_i G$, consequently according to (96) $V' = 0$. In other words, Nurkse imbalance would be zero.

Applying to the same period and the same countries as Figure 2, our Figure 3 shows that Nurkse imbalance is far from zero. The

Nurkse imbalance in Figure 3 is almost as substantial as the Samuelson-Solow-von Neumann imbalance in Figure 2. The Nurkse range has the same floor but a slightly lower ceiling than the Samuelson-Solow-von Neumann range: The square root of the weighted sum of squared Nurkse deviations ranges from 0.19 (the Netherlands) to 1.0 (Uruguay) of the national real growth rate, with a majority of the countries lying between 0.25 and 0.50 of that rate. We conclude that the Nurkse index has removed precious little imbalance from the Samuelson-Solow-von Neumann index.

How come, so little? Suppose all sector income elasticities were unity, then the Samuelson-Solow-von Neumann index would become equal to the Nurkse index: If $E_i = 1$ it follows from (95) and (96) that $V^* = V'$. And indeed the income elasticities used by Yotopoulos-Lau differed very little from unity:

Agriculture	0.952
Mining	0.892
Manufacturing	1.044
Construction	1.035
Electricity-gas-water	1.045
Others	0.999

These sector income elasticities were estimated from cross sections of some of the countries examined but applied to all countries.

From the Yotopoulos-Lau measurements we conclude three things. First, that growth imbalance is a rather ubiquitous phenomenon. Second, that in highly developed countries it is not strongly correlated with the national real growth rate. Third, that nonunitary sector income elasticities play a minuscule role in explaining real-world growth imbalance.

F O O T N O T E S

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[1]We define, as Hahn and Matthews [3] did, steady-state growth as stationary proportionate rates of growth of physical outputs. We define, as Solow and Samuelson [4] did, balanced growth as identical proportionate rates of growth of physical output for all goods.

[2]Our Graham-type demand functions (38) and (39) have unitary income elasticities. In our model, then, possible growth imbalance must have causes other than nonunitary income elasticities. From Yotopoulos-Lau [6] one may conclude that nonunitary sector income elasticities play a minuscule role in explaining real-world growth imbalance. This conclusion is derived in Appendix III.

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GREEK LETTERS USED

α	alpha
β	beta
ζ	zeta
κ	kappa
λ	lambda
μ	mu
ν	nu
ξ	xi
π	pi
ρ	rho
Σ	sigma
τ	tau
ϕ	phi
ψ	psi
ω	omega

MATHEMATICAL SYMBOLS USED

$\{\ \}$	brace
$[\ ]$	bracket
$\lvert\ \rvert$	determinant
$=$	equal to
$>$	greater than
$\equiv$	identically equal to
$\int$	integral of
$<$	less than
$\neq$	not equal to
$(\ )$	parenthesis
∂	partial derivative of
$\surd$	square root of